chap-1 Introduction to plasma Physics by F.F. Chen. ①

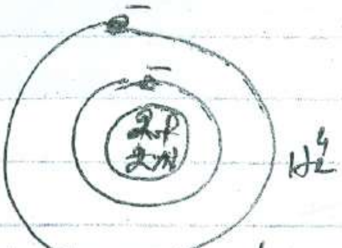
Dr. Yaqub.

Occurrence of Plasma in Nature

Plasma is fourth state of matter.

Solid → liquid → gas → plasma

99% of the matter in universe is in the form of plasma.



this is 4th state of matter.

If we supply heat energy to the gas molecules than outermost electrons will escape ~~after absorbing energy~~. If we continue to suply heat energy than all the atoms will dissociate into positive ions and negative electrons

① The stellar (stars) interiors, atmospheres, gaseous nebulae are plasmas, In the vicinity of our atmospher ③ van Allen radiation belts and the solar wind are plasmas.

While one can find on earth the flash of a lightening bolt, the soft glow of the Aurora Borealis, the conducting gas inside of a fluores-cent tube etc

~~① S~~ ② a cloud of dust, hydrogen gas, helium gas and other ionized gases ③ is a torus of energetic charged particles around

Saha equation

It tells us the amount of ionization to be expected in a gas in thermal equilibrium.

$$\frac{n_i}{n_n} \approx 2.4 \times 10^{21} \frac{T^{3/2}}{n_i} e^{-U_i/KT} \quad \text{--- (1)}$$

... density of

Saha Equation:

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$$\frac{n_i}{n_n} \approx 2.4\times10^{21}\,\frac{T^{3/2}}{n_i}\,e^{-U_i/KT} \quad \text{--- (1)}$$

n_i and n_n are the number density of ionized atoms and of neutral atoms, T is gas temperature in K°, K is Boltzmann's constant and U_i is the ionization energy of the gas.

At room temperature

$n_n \approx 3\times10^{25}\,m^{-3}$, $T = 300\,^\circ K$

$U_i = 14.5\,eV$ for nitrogen

$1\,eV = 1.6\times10^{-19}\,J$

thus eqn. (1) gives

$$n_i \sim 10^{-122}$$

As the temp. is raised, the degree of ionization remains low until U_i is only a few times KT. Then $\frac{n_i}{n_n}$ rises abruptly and the gas is in a plasma state. Further increase in temp. makes n_n less than n_i and plasma eventually becomes fully ionized.

$^{1}_{0}n + ^{235}U \rightarrow ^{236}U \rightarrow ^{92}Kr + ^{141}Ba + 3n$

0.9MeV energy per nucleon.

ε_0 - permittivity = is the measure of the resis
that is encountered when forming an electric field in a me

Definition of Plasma:

A plasma is a quasineutral gas of charged and neutral particles which exhibits collective behavior.

Quasineutrality condition is:
number density of electrons
$\approx$ no. density of ions

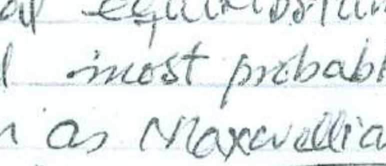
$$n_e \approx n_i$$

If $n_e \neq n_i$ then there exists electric field which is $\vec{\nabla} \cdot \vec{E} = 4\pi \rho_e$ (Poisson's Eqn)

$$= 4\pi e(n_i - n_e)$$

Collective behavior:

The state of plasma in regions at some distances from the point of interest may affect the behavior.



Concept of Temperature:

A gas in "thermal equilibrium has particles of all velocities and most probable distribution of these velocities is known as Maxwellian dist.

$$f(u) = A \exp(-\tfrac{1}{2} m u^2 / KT)$$

also called probability Distb. fun

Thus $n = \int_{-\infty}^{\infty} f(u)\,du$ is density n, or no. of particles per cm^3.

[To speify the position of a particle in (N) we need $(\bar{r}, t, \bar{p})$ 7 coordinates this is called <u>phase space</u>].

$$n = A\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{mu^2}{KT}}\, du$$

$\alpha = \frac{m}{2KT}$

we know that $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi/\alpha}$

$$\therefore \quad n = A \sqrt{\pi / \tfrac{m}{2KT}}$$



$$\Rightarrow \quad A = n\sqrt{\frac{m}{2KT\pi}}$$

~~this is~~ $\Rightarrow \quad f(u) = n\sqrt{\frac{m}{2\pi KT}}\; e^{-\frac{1}{2}\frac{mu^2}{KT}}$

is the exact form of Maxwellian velocity distribution.

~~Consider~~ Averaging

$$\left\langle \tfrac{1}{2} mu^2 \right\rangle = \frac{\int_{-\infty}^{\infty} f(u)\,(\tfrac{1}{2}mu^2)\, du}{\int_{-\infty}^{\infty} f(u)\, du}$$

$$= \frac{\int_{-\infty}^{\infty} n\sqrt{\frac{m}{2\pi KT}}\, e^{-\frac{1}{2}\frac{mu^2}{KT}} \cdot (\tfrac{1}{2}mu^2)\, du}{\int_{-\infty}^{\infty} n\sqrt{\frac{m}{2\pi KT}}\, e^{-\frac{1}{2}\frac{mu^2}{KT}}\, du}$$

Let $y = \sqrt{\frac{m}{2KT}}\, u$

$$u = \sqrt{\frac{2KT}{m}}\, y.$$

$$du = \sqrt{\frac{2KT}{m}}\, dy.$$

$$\therefore \langle \tfrac{1}{2} m u^2 \rangle = \frac{\frac{1}{2} m \int_{-\infty}^{\infty} e^{-y^2} y^2 dy \cdot \frac{2KT}{m} \times \sqrt{\frac{2KT}{m}}}{\int_{-\infty}^{\infty} e^{-y^2} \sqrt{\frac{2KT}{m}}\, dy}.$$

$$= \frac{1}{2} m \cdot \frac{2KT}{m} \frac{\int_{-\infty}^{\infty} e^{-y^2} y^2 dy}{\int_{-\infty}^{\infty} e^{-y^2} dy}.$$

$$= KT \frac{\int_{-\infty}^{\infty} e^{-y^2} y^2 dy}{\int_{-\infty}^{\infty} e^{-y^2} dy}. \quad \text{—— (1)}$$

Consider $\int_{-\infty}^{\infty} e^{-y^2} y^2 dy = -\frac{1}{2}\int_{-\infty}^{\infty} y \cdot e^{-y^2} (-2y\, dy).$

$$\frac{y}{e^{y^2}} \to 0$$

$$\frac{1}{2.72}, \frac{2}{55}, \frac{3}{8103} \to 0$$

$$= -\frac{1}{2}\left[y\, e^{-y^2} \Big|_{-\infty}^{\infty} \to 0 - \int_{-\infty}^{\infty} e^{-y^2} dy \right]$$

$$= +\frac{1}{2} \int_{-\infty}^{\infty} e^{-y^2} dy$$

Thus (1) $\Rightarrow$

$$\langle \tfrac{1}{2} m u^2 \rangle = \frac{KT \cdot \frac{1}{2}\int_{-\infty}^{\infty} \bar{e}^{y^2} dy}{\int_{-\infty}^{\infty} \bar{e}^{y^2} dy} = \tfrac{1}{2} KT.$$

Thus Average Kinetic energy is $\frac{1}{2}$ KT.

No (2)

In 3-dimensional space,

$$\langle \tfrac{1}{2} m (u^2+v^2+w^2) \rangle = ?$$

$$= \frac{\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dw \left(\tfrac{1}{2} m (u^2+v^2+w^2)\right) A e^{-\frac{1}{2} m \frac{(u^2+v^2+w^2)}{KT}}}{\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dw \, A e^{-\frac{1}{2} m \frac{(u^2+v^2+w^2)}{KT}}}$$

for u^2 only

$$= \frac{\int_{-\infty}^{\infty} \tfrac{1}{2} m u^2 e^{-\frac{1}{2}\frac{mu^2}{KT}} du \int e^{-\frac{1}{2}\frac{mv^2}{KT}} dv \int e^{-\frac{1}{2}\frac{mw^2}{KT}} dw}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{mu^2}{KT}} du \int e^{-\frac{1}{2}\frac{mv^2}{KT}} dv \int e^{-\frac{1}{2}\frac{mw^2}{KT}} dw}$$

$$= \tfrac{1}{2} KT$$

Similarly for v^2 and w^2, we get

$$\langle \tfrac{1}{2} m (u^2+v^2+w^2) \rangle = \tfrac{1}{2} KT + \tfrac{1}{2} KT + \tfrac{1}{2} KT$$

$$= \tfrac{3}{2} KT$$

$$\bar{\nabla} \times \bar{V} = \left(\frac{1}{s} \frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z} \right) \hat{s} + \left(\frac{\partial V_s}{\partial z} - \frac{\partial V_z}{\partial s} \right) \hat{\phi}$$

$$+ \frac{1}{s} \left[\frac{\partial}{\partial s} (s V_\phi) - \frac{\partial V_s}{\partial \phi} \right] \hat{z} .$$

in $\hat{r}, \hat{\theta}, z$ are

$$\bar{\nabla} \times \bar{V} = \left(\frac{1}{r} \frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z} \right) \hat{r} + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} \right) \hat{\theta}$$

$$+ \frac{1}{r} \left[\frac{\partial}{\partial r} (r V_\theta) - \frac{\partial V_r}{\partial \theta} \right] \hat{z} .$$

1 eV ~ temp?

$$KT = 1\,eV$$

$$T = \frac{1\,eV}{K} = \frac{1.6\times10^{-19}\,erg}{1.38\times10^{-23}\,erg/K}$$

$$T \approx 11600\,K$$

$$1000\,eV = 10^{7}\,°K$$

$$K\,eV = 10^{7}\,°K$$

Debye Shielding

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}.$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Debye Shielding

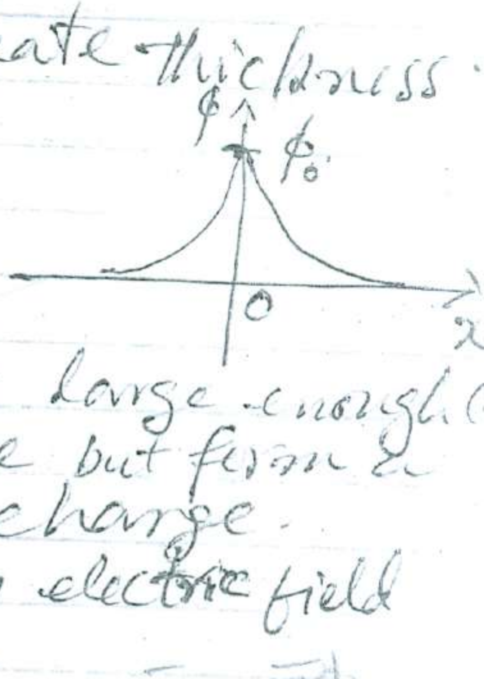
~~An impor~~

A fundamental characteristic of plasma is to shield out electric potentials that are applied to it. Suppose we tried to put an electric field inside a plasma by inserting two charged balls connected to a battery. The balls would attract particles of the opposite charge, and almost immediately a cloud of ions would surround the negative ball and a cloud of electrons would surround the positive balls.



(8)

If the plasma were cold and there were no thermal motions, there would be just as many charges in the cloud as in the ball. The shielding would be perfect and no electric field would be present in the ~~ball~~ body of the plasma. At the edge of the cloud potential energy is approximately equal to the thermal energy KT of the particles.
Radius of sphere =
Let us compute the approximate thickness of such a charge cloud.
Consider the fig.,



$\phi(x) = ?$
Here we assume that M/m is large enough such that the ions do not move but form a uniform background of positive charge.
If $n_i \neq n_e$ then electric field is established.

$$\nabla \cdot \bar{E} = 4\pi e(n_i - n_e) \qquad E = -\nabla\phi$$

and Poisson's Equation

$$\nabla^2\phi = -4\pi e(n_i - n_e)$$

As due to spherical symmetry of this electrostatic potential, so we can restrict in only one dimension.

$$\frac{d^2\phi}{dx^2} = -4\pi e(n_i - n_e)$$

Let

number density of electrons

$$f(u) = A\, e^{-(\frac{1}{2}mu^2 + q\phi)/KT}.$$

is Maxwell-Boltzmann Distribution.

$$n_e = \int_{-\infty}^{\infty} f(u)\,du = A\int_{-\infty}^{\infty} e^{-\frac{1}{2}mu^2/KT}\, e^{-q\phi/KT}\,du$$

$$= n\sqrt{\frac{m}{2\pi KT}}\, e^{-q\phi/KT} \int_{-\infty}^{\infty} e^{-\frac{1}{2}mu^2/KT}\,du \qquad \left| A = \sqrt{\frac{m}{2KT}} \right.$$

for e $n_e = n\sqrt{\frac{m}{2\pi KT}} \times \sqrt{\frac{2\pi KT}{m}}\, e^{-q\phi/KT} = n\, e^{-q\phi/KT}$

$$n_e = n_\infty e^{-q\phi/KT} = n_\infty e^{e\phi/KT}$$

$$\Rightarrow \frac{d^2\phi}{dx^2} = -4\pi e n_\infty \left(1 - e^{e\phi/KT}\right) \qquad \boxed{q = -e}$$

$$= 4\pi e n_\infty \left[\frac{e\phi}{KT} + \frac{1}{2!}\left(\frac{e\phi}{KT}\right)^2 + \cdots \right]$$

As $e\phi << KT$ ~~So the~~ can

$$\Rightarrow \frac{d^2\phi}{dx^2} = 4\pi e n_\infty \frac{e\phi}{KT}$$

$$\Rightarrow \frac{d^2\phi}{dx^2} = \frac{4\pi e^2 n_\infty}{KT}\phi.$$

let $\dfrac{d^2\phi}{dx^2} = \dfrac{1}{\lambda_D^2}\phi$; $\lambda_D = \left(\dfrac{KT_e}{4\pi e^2 n}\right)^{1/2}$

λ_D is called Debye length, is a measure of the shielding distance or thickness of the sheath.

$$\Rightarrow \quad \phi = A e^{-x/\lambda_D} + B e^{x/\lambda_D}.$$

For finite potential $B e^{x/\lambda_D}$ is unphysical.
Therefore,

$$\phi = A e^{-x/\lambda_D}$$

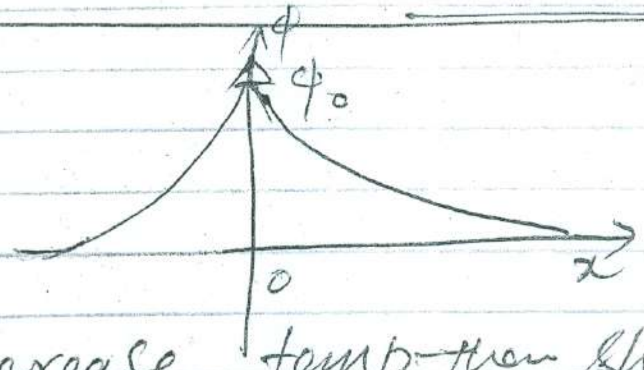
or

$$\phi = \phi_0 e^{-|x|/\lambda_D}$$

This is called Debye Potential.

(+) ——— r ———→ $\quad \phi = q/4\pi r$

Coulomb Potential.



$$\lambda_D = \left(\frac{KT_e}{4\pi e^2 n}\right)^{1/2}$$

If we increase temp. then shielding or size of Debye sphere is increased.
If we increase no. density of particles the size of sphere is decreased.
The useful forms of λ_D are,

$$\lambda_D = 69 \left(T/n\right)^{1/2} m, \quad T \text{ in } k$$

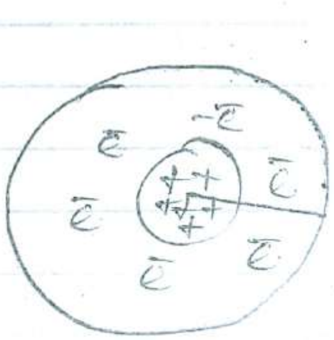
~~Consider~~ $\lambda_D = \left(\frac{KT_e}{4\pi n e^2}\right)^{1/2}$

$= 69 \left(\frac{T(K)}{n_\infty\, cm^{-3}}\right)^{1/2} cm$

An erg is the unit of work. An erg is the amount of work done by a force of one dyne exerted for a distance of one centimeter.

Lec# 03

Plasma Parameter



We know that,

$$V = \frac{4}{3}\pi r^3$$

and for Debye Sphere

$$V = \frac{4}{3}\pi \lambda_D^3$$

So number of particles in the sphere is

$$N_D = n \cdot \frac{4}{3}\pi \lambda_D^3$$

In plasma state for shielding there should be large no of particles approx $N_D >>> 1$

$n_i \sim n_e$.

also $\lambda_D << L$

$$\lambda_D = \left(\frac{KT_e}{4\pi e^2 n}\right)^{1/2}$$

λ_D .

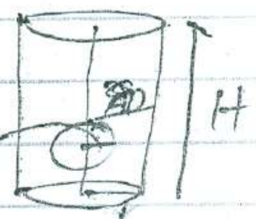
① parameters – T_e
② no. density – n
③ we will make (e. as Third.)

1 Dyne = 10 μN
The force required to accelerate one gm of mass at a rate 1 cm/sec²

Criteria for Plasma

There are three conditions, a plasma must satisfy,

(1) $\lambda_D << L$



Suppose the plasma in a cylinder the Debye Sphere ~~should~~ be λ_D inside the cylinder

λ_D ~~must com~~ should be compared with smallest dimension such as "L" not "H"!
in the lab.

(2) $N_D >>> 1$

as $N_D = \frac{4}{3}\pi n \lambda_D^3$

For shielding number of particles should be in large number.

~~appro~~ ~~e.g.~~ $n_i \approx n_e$

(3) $\omega\tau >> 1$



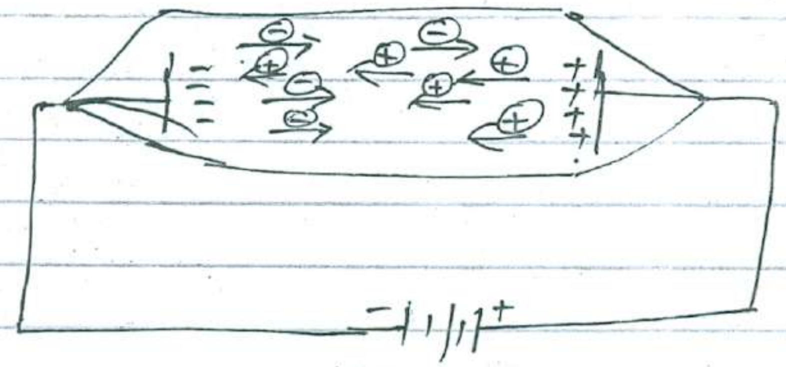
In the cylinder ionized gas is filled, if ω is frequency of the plasma particles ($\bar{e}$, ⊕) and τ is mean time between collisions with neutral particles, then, it should follow, if it is plasma

$\omega >> \tau^{-1}$ or

$\omega\tau >> 1$

Applications of Plasma.

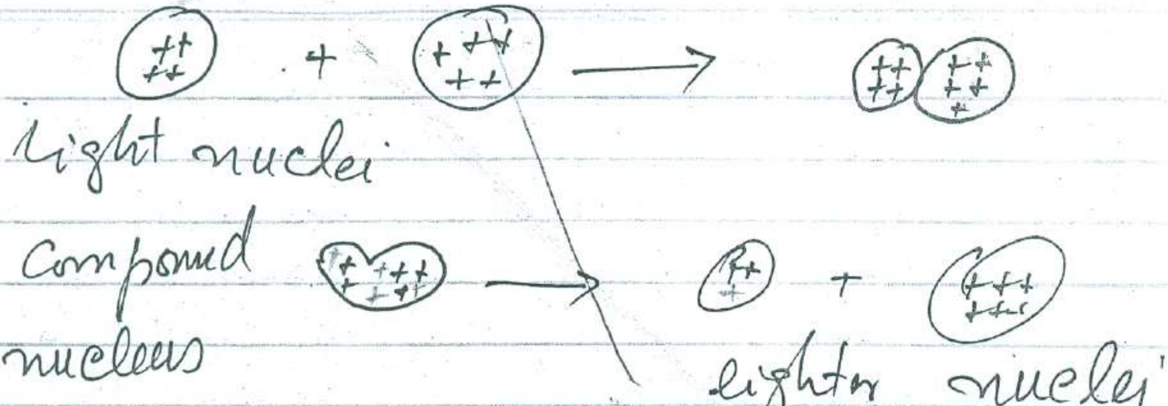
(1) Gas Discharge.



In 1920, Langmuir, Tonks and their collaborators ~~developed~~ worked on vacuum tubes with weakly ionized gases, and positive column with $kT_e = 2eV$ also $10^{14} < n < 10^{18}$ /m³. Shielding phenomena was discovered and sheath surrounding an electrode could be seen as a dark layer. Some examples are mercury rectifiers, hydrogen thyratron, etc.

(2) Controlled thermonuclear Fusion.

Modern plasma physics had its beginnings around 1952, when it was proposed that the hydrogen bomb fusion reaction be controlled to make a reactor. ~~Some reactions are~~



For fusion scientists use isotopes of hydrogen not the hydrogen.

$^{2}_{1}D + ^{2}_{1}D \longrightarrow {}^{3}He + n + 3.2\ MeV.$

$^{2}D + {}^{2}D \longrightarrow T + P + 4.0\ MeV.$

$^{2}D + {}^{3}T \longrightarrow {}^{4}He + n + 17.6\ MeV.$

It is necessary to create a plasma in which the thermal energies are in the 10-keV range. The problem of heating and containing such a plasma is responsible for the rapid growth of the science of plasma but the problem is still unsolved.

L#04: Motion of Single charged Particles.

Plasma behaves sometimes like fluids and some times like a collection of individual particles. Here we will see the behavior of single particle in electric and magnetic fields.

case-(I)

when $\bar{E} = 0$.

We will discuss the motion of charged particles in the presence of magnetic field only. Consider the equation of motion, Lorentz force

$F = q \bar{v} \times \bar{B}$

$F = q(E$

$$m \frac{d\bar{v}}{dt} = q(\bar{v} \times \bar{B})$$

For simplicity we take $\bar{B} = B\hat{z}$

$$\Rightarrow m \frac{d\bar{v}}{dt} = q \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix}$$

$$m(\dot{v}_x \hat{i} + \dot{v}_y \hat{j} + \dot{v}_z \hat{k}) = q\{\hat{i} B v_y - \hat{j} v_x B + \hat{k} \times 0\}$$

$$\left.\begin{aligned} \Rightarrow m\dot{v}_x = qBv_y \quad &, \quad m\dot{v}_y = -qBv_x \\ m\dot{v}_z &= 0 \end{aligned}\right\} \quad (1)$$

Diff. w.r.t "t",

$$\Rightarrow m\ddot{V}_x = qB\dot{V}_y = qB\left(-\frac{qB}{m}\right)V_x$$

~~$m\ddot{V}_y = -qB\dot{V}_x$~~

$$m\ddot{V}_x = -\frac{q^2B^2}{m}V_x$$

$$\ddot{V}_x = -\frac{q^2B^2}{m^2}V_x$$

$$\ddot{V}_x = -\left(\frac{qB}{m}\right)^2 V_x$$

Similarly $\ddot{V}_y = -\left(\frac{qB}{m}\right)^2 V_y$ and $\dot{V}_z = 0$.

Let $\omega_c = \frac{|q|B}{m}$ as cyclotron frequency

<u>cyclotron frequency</u> = The frequency of rotation of a charged particle inside a homogeneous magnetic field.

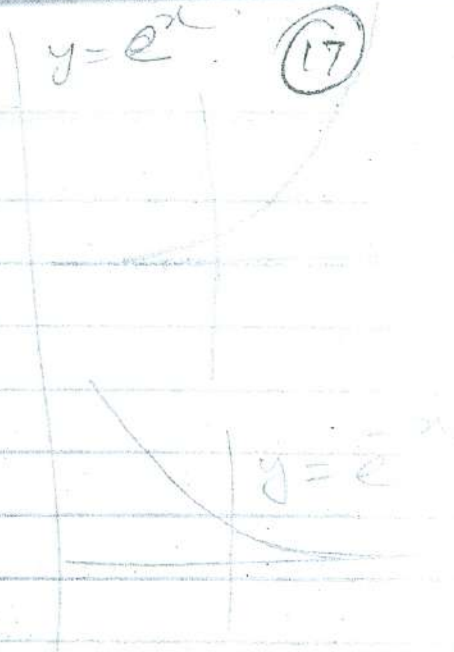
Thus we have

$$\ddot{V}_x = -\omega_c^2 V_x \quad \text{As } \dot{V}_z = 0$$

$$\ddot{V}_y = -\omega_c^2 V_y$$

$$\dot{V}_z = 0 \Rightarrow V_z = \text{constant}.$$

$\Rightarrow$ z-component of velocity is constant!



Thus $V_x = V_\perp e^{i\omega_c t}$

~~and $V_y = iV_\perp e^{i\omega_c t}$~~

$$\Rightarrow \quad \dot{V}_x = i V_\perp \omega_c e^{i\omega_c t}$$

Then $V_y = \frac{m}{qB} \dot{V}_x$

$$V_y = \frac{1}{\omega_c} \dot{V}_x$$

$$= \frac{1}{\omega_c} i V_\perp \omega_c e^{i\omega_c t}$$

$$V_y = i V_\perp e^{i\omega_c t}$$

[$e^{i\theta} = \cos\theta + i\sin\theta$
$ie^{i\theta} = -\sin\theta + i\cos\theta$]

Considering only the real part we have,

$$V_x = \dot{x} = V_\perp \cos\omega_c t$$

$$V_y = \dot{y} = -V_\perp \sin\omega_c t \quad \text{—— (*)}$$

Thus $x = V_\perp \int \cos\omega_c t \, dt = V_\perp \frac{\sin\omega_c t}{\omega_c} + x_0$

and $y = -V_\perp \int \sin\omega_c t \, dt = +V_\perp \frac{\cos\omega_c t}{\omega_c} + y_0$

$\Rightarrow x - x_0 = \frac{v_\perp}{\omega_c} \sin \omega_c t$

$y - y_0 = + \frac{v_\perp}{\omega_c} \cos \omega_c t$

Squaring and adding

$(x - x_0)^2 + (y - y_0)^2 = \frac{v_\perp^2}{\omega_c^2} \{ \sin^2 \omega_c t + \cos^2 \omega_c t \}$

$\Rightarrow (x - x_0)^2 + (y - y_0)^2 = \left(\frac{v_\perp}{\omega_c}\right)^2$

This is equation of a circle whose centre is (x_0, y_0) and radius

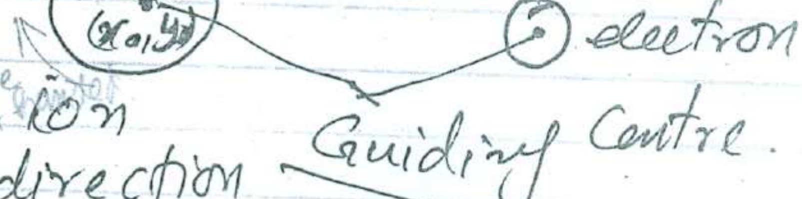
$r_L = \frac{v_\perp}{\omega_c} = \frac{v_\perp m}{|q| B}$; r_L = Larmour radius.

Energy of the part. constant,

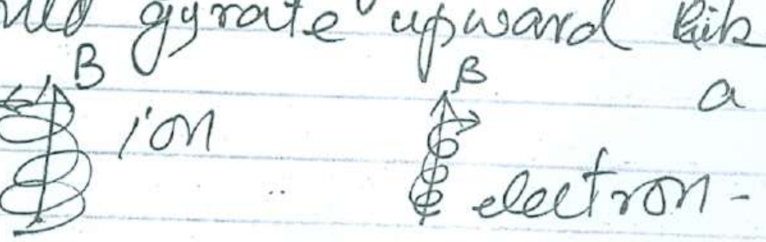
$m \frac{dv}{dt} = q(\bar{v} \times \bar{B})$.

$m \bar{v} \cdot \frac{d\bar{v}}{dt} = q \, \bar{v} \cdot (\bar{v} \times \bar{B})$

$\Rightarrow \frac{d}{dt}\left(\frac{1}{2} m v^2\right) = 0$



As velocity in z-direction is constant, therefore if we see from the top that it would gyrate upward like ~~a~~ ~~path~~ a helix.



Thus we see the motion of a charged particle in a uniform B is circular with a guiding centre (x_0, y_0). The direction of the gyration is always such that the magnetic field generated by the charged particle is opposite to the externally applied field. In the result plasma particles tend to reduce the magnetic field and plasmas are diamagnetic.

C - b Finite $\bar{E}$

If we allow an electric field to be present, the motion will be found to be the sum of two motions, the usual circular Larmor gyration plus a drift of the guiding centre. We may choose E to lie in the xz-plane so that $E_y = 0$. The eqn. of motion is now,

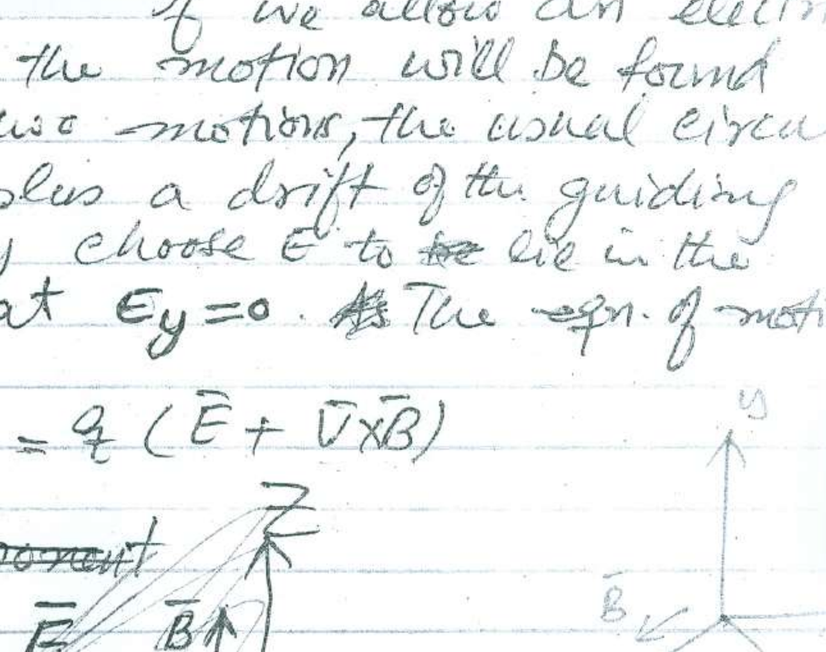
$$m\frac{d\bar{v}}{dt} = q(\bar{E} + \bar{v} \times \bar{B})$$

~~The z-component~~

In $E = 0$ $V_{\parallel}$ = constant.

Clearly, here, $\dot{V}_{\parallel} = \frac{q\bar{E}}{}$

constant acceleration along the field



$$m\frac{dV_x}{dt} = q(E_x + V_y B)$$

$$m\frac{dV_y}{dt} = q(0 - V_x B)$$

$$m\frac{dV_z}{dt} = qE_z$$

$$\int \frac{dV_z}{dt}\, dt = \frac{qE_z}{m}\int dt$$

$$\boxed{V_z = \frac{qE_z}{m}t + V_0}$$

$$\dot{V}_x = \frac{qE_x}{m} + \frac{qB}{m}V_y$$

$$\dot{V}_x = \frac{qE_x}{m} \pm \omega_c V_y$$

$$\dot{V}_y = -\frac{qB}{m}V_x$$

$$\dot{V}_y = \mp \omega_c V_x$$

$$\ddot{V}_x = \pm \omega_c \dot{V}_y$$

$$\ddot{V}_x = \pm \omega_c(\mp \omega_c V_x)$$

$$\ddot{V}_x = -\omega_c^2 V_x$$

$\frac{d}{dt}E_x = 0$

E is not time dependent

and $\ddot{V}_y = \mp \omega_c \dot{V}_x$

$$= \mp \omega_c \left(\frac{q}{m} E_x \pm \omega_c V_y \right)$$

$$\ddot{V}_y = -\frac{q\,\omega_c}{m} E_x - \omega_c^2 V_y$$

$$= -\omega_c^2 \left[V_y + \frac{q}{m\omega_c} E_x \right]$$

$$\ddot{V}_y = -\omega_c^2 \left[V_y + \frac{E_x}{B} \right] \quad \text{— (a)}$$

Let $V_y = v_y + E_x / B$

$$v_y = V_y - \frac{E_x}{B}$$

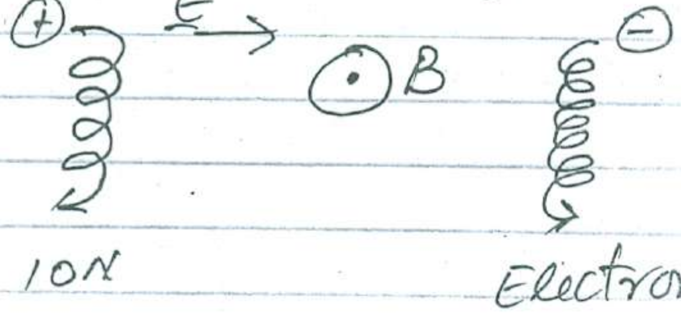
$$\dot{v}_y = \dot{V}_y$$

$$\ddot{v}_y = \ddot{V}_y$$

(a) $\Rightarrow$ $\ddot{v}_y = -\omega_c^2 v_y$

$$\left.\begin{aligned} V_y &= -V_\perp \sin\omega_c t \\ V_x &= V_\perp \cos\omega_c t \end{aligned}\right] \text{— I}$$

In this case the Larmor motion is the same as before but there is superimposed a drift V_{gc} of the guiding centre in the y-direction. ~~(for $E_x > 0$)~~



To obtain a general formula for V_{gc}, we may omit the $m\frac{dv}{dt}$ term, since this term gives only the circular motion at ω_c.

Thus we have

$$m\frac{d\bar{v}}{dt} = q(\bar{E} + \bar{V}\times\bar{B})$$

$$\Rightarrow \quad \bar{E} + \bar{V}\times\bar{B} = 0$$

cross product with $\bar{B}$ gives,

$$\Rightarrow \quad \bar{E}\times\bar{B} + (\bar{V}\times\bar{B})\times\bar{B} = 0$$

$$\Rightarrow \quad \bar{E}\times\bar{B} = \bar{B}\times(\bar{V}\times\bar{B})$$

$$= \bar{V}B^2 - \bar{B}(\bar{V}\cdot\bar{B})$$

$\bar{B}\cdot\bar{V}_\perp = 0$ for $\bar{V} = \bar{V}_\perp$

$$\bar{E} \times \bar{B} = \bar{V}_\perp B^2 - 0$$

$$\Longrightarrow \boxed{\bar{V}_\perp = \frac{\bar{E} \times \bar{B}}{B^2}} = \frac{1}{B^2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ E_x & 0 & E_z \\ 0 & 0 & B \end{vmatrix}$$

Thus we have

$$\bar{V}_\perp = - \frac{E_x B}{B^2} \hat{j}$$

$$\bar{V}_\perp = V_E = - \frac{E_x}{B} \hat{j}$$

It is important to note that $\bar{V}_E$ is independ of q, m and $V_\perp$. The reason is obvious because, in the ~~half as~~ first half-cycle of the ion's orbit, it gains energy from the electric field and increases in $V_\perp$ and hence in r_L. In the second half-cycle, it loses energy and decreases ~~of~~ r_L. This difference in r_L (~~along~~ opposite and along ~~opposite to~~ E) on the left and right sides of the orbit causes the drift V_E.

A negative electron gyrates in the opposite direction but also gains energy in the opposite direction. Its drift is also in the same direction as ion.

For particles of the same velocity but different mass, the lighter one will have smaller r_L

and hence drift less per cycle. Its gyration frequency is also larger and the two effects exactly cancel.

Two particles of the same mass but different energy would have the same ω_c. The slower one will have smaller r_L and hence gain less energy from $\bar{E}$ in a half-cycle.

Gravitational Field.

We know that

$$\bar{V}_E = \frac{\bar{E}\times\bar{B}}{B^2} \Rightarrow \bar{V}_E = q\,\frac{\bar{E}\times\bar{B}}{qB^2}$$

$$\therefore \quad \bar{V}_F = \frac{\bar{F}\times\bar{B}}{qB^2}$$

So in the presence of gravity

$$\bar{F} = m\bar{g}$$

$$\Rightarrow \bar{V}_g = \frac{m\,\bar{g}\times\bar{B}}{qB^2}$$

current density

$\bar{J} = \rho(\bar{r},t)\, V_d(\bar{r},t)$

$= q\, n(\bar{r},t)\, V_d$

$\bar{J} = \sigma E$

of charges distribution flowing at a point at time t

This drift $\bar{V}_g$ changes sign with the particle charge. Under a gravitational force ions and electrons drift in opposite direction, so there is a net current density in the plasma,

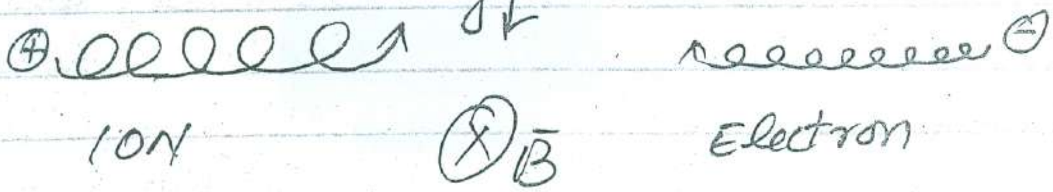
$$\bar{J} = e n_i \bar{V}_i - e n_e \bar{V}_e$$

using quasi-neutrality condition $n_i \approx n_e$. and $V = \frac{m}{q}\frac{\bar{g}\times B}{B^2}$

$$\Rightarrow \bar{J} = en(\bar{V}_i - \bar{V}_e)$$

$$= en\left(\frac{M}{e} + \frac{m}{e}\right)\frac{\bar{g}\times\bar{B}}{B^2}$$

$$\bar{J} = n(M+m)\frac{\bar{g}\times\bar{B}}{B^2}$$



So $\bar{J} \neq 0$.

The physical reason for this drift is again the change in Larmor radius as the particle gains or loses energy in the gravitational field. Now the electrons gyrate in the opposite sense to the ions. The magnitude of $\bar{V}_g$ is negligible but when the lines of force are curved. There is an effective gravitational force due to Centrifugal force, which is responsible of "gravitational instability."

Tue - 06/03/12

Non-Uniform $\bar{B}$ - Field.

(a) $\nabla B \perp \bar{B}$



In this case, we take the lines of force as straight but their density increases, in y-direction, the ~~gradient~~ ∇B causes the Larmor radius to be larger at the bottom of the orbit than at the top and this should lead to a drift in opposite directions for ions and electrons perpendicular to both $\bar{B}$ and ∇B.

$$\bar{F} = q\,\bar{v} \times \bar{B}$$

$$\Rightarrow \quad F_x = q\, v_y B$$

$$F_y = -q\, v_x B$$

$$F_z = 0$$

K.E is the energy ~~[illegible]~~ which it possesses due to its motion & work needed to accelerate a body of a given mass from rest to its stated velocity

$$\bar{F_x} = \frac{1}{T}\int_0^T F_x\, dt$$

$$F_y = -q\,\bar{v}_x B$$

$$= -q\, v_\perp \cos\omega_c t\; B(y).$$

By Taylor's expansion: here $r_L \ll \frac{B}{|\nabla B|}$

so we can express $B(y)$,

$$F_y = -qV_\perp \cos\omega_c t\left[B_0 \pm y\frac{\partial B}{\partial y} + \cdots\right].$$

for $y - y_0 = r_L \cos\omega_c t$

let $y = r_L \cos\omega_c t$ for $y_0 = 0$

$$F_y = -qV_\perp \cos\omega_c t\left[B_0 \pm r_L \cos\omega_c t \frac{\partial B}{\partial y} + \cdots\right]$$

~~$B = B_0 + (r\cdot\nabla)B + \frac{r^2\nabla^2 B}{2!}$~~

$= B_0 + y\frac{\partial B}{\partial y} + \cdots$

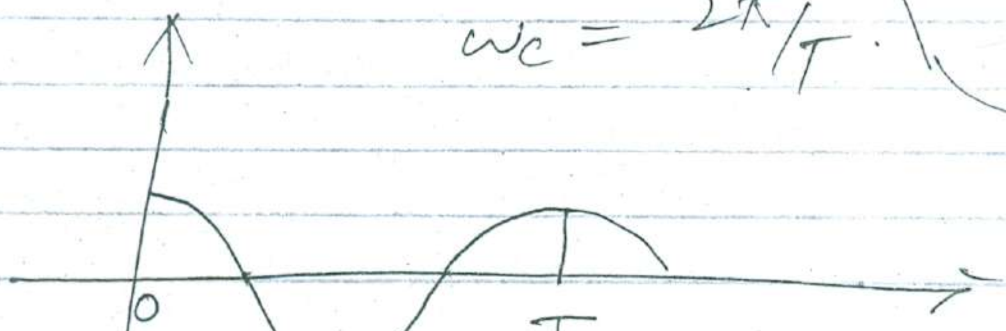
$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

$a = 0,\ x = y.$

and $\frac{1}{T}\int_0^T \cos\omega_c t\, dt = 0$

$\omega_c = 2\pi f$

$\omega_c = 2\pi/T.$



$$\Rightarrow \bar{F}_y = -q\frac{V_\perp r_L}{2}\frac{\partial \bar{B}}{\partial y}$$

As choice of y-axis was arbitrary, we may generalize it

Thus $\bar{V}_F = \frac{F\times B}{qB^2} = -\frac{qV_\perp r_L}{2}\frac{\nabla B\times B}{qB^2}$

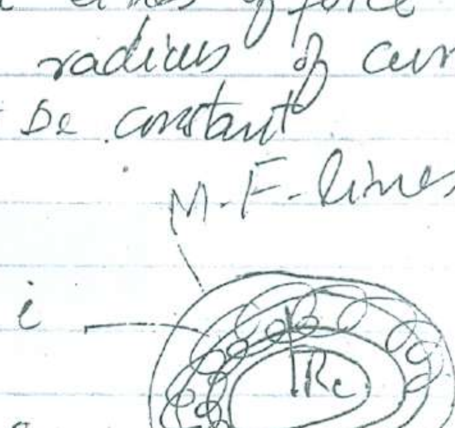
$$\bar{V}_{\nabla B} = \frac{1}{2}V_\perp r_L \frac{B\times\nabla B}{B^2}$$

This is called grad-B drift, it causes a current transverse to B.

\#06

Curved $\bar{B}$ Curvature Drift.

Here we assume the lines of force to be curved with a constant radius of curvature R_c and we take $|B|$ to be constant.



(9) According to Ampere's Law

$$\nabla \times B = \frac{4\pi}{c} J = 0$$

As there is vacuum in the toroid therefore $J = 0$

For vacuum, $(\bar{\nabla} \times \bar{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(r B_\theta) = 0$

$$\Rightarrow B_\theta = \frac{\text{constant}(c_1)}{r}$$

$$\Rightarrow B \propto \frac{1}{r} \qquad B = \frac{\text{const}(c_1)}{r}$$

Thus $\frac{\bar{\nabla} B}{B} = \frac{\text{const}/r^2}{\text{const}/r}\hat{r} = -\frac{\hat{r}}{r}$

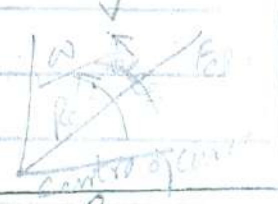
If $\bar{R}_c$ represents position vector of any magnetic field line, then, $\frac{\bar{\nabla} B}{B} = -\frac{\bar{R}_c}{R_c^2}$

(V_E)

For the case of curvature ~~of~~ field lines we must add $v_{\nabla B}$.

I. If $v_{\parallel}^2$ is the average square of the component of random velocity along B, the average centrifugal force is

$$\bar{F}_{cf} = \frac{m v_{\parallel}^2}{R_c}\hat{r} = m v_{\parallel}^2 \frac{\bar{R}_c}{R_c^2}$$



Thus

$$\bar{V}_R = \frac{1}{q}\frac{\bar{F}_{cf}\times\bar{B}}{B^2}$$

$$\bar{V}_R = \frac{m v_{\parallel}^2}{qB^2}\frac{\bar{R}_c\times\bar{B}}{R_c^2}$$

The curvature drift.

Pitch angle = angle between vector velocity of charged particle and local magnetic field.

Thus using

$$V_{\nabla B} = \frac{1}{2} v_{\perp} r_L \frac{\bar{B}\times\nabla\bar{B}}{B^2}$$

$$\Rightarrow \quad = -\frac{1}{2} v_{\perp} r_L \frac{B\times|B| R_c}{B^2 R_c^2}$$

$$= \frac{1}{2}\frac{v_{\perp}^2}{\omega_c}\frac{R_c\times\bar{B}}{R_c^2 B}$$

$$r_L = \frac{v_{\perp}}{\omega_c}, \quad \omega_c = \frac{qB}{m}$$

$$\bar{V}_{\nabla B} = \frac{1}{2}\frac{m}{q} v_{\perp}^2 \frac{\bar{R}_c\times B}{R_c^2 B^2}$$ ✓

I.

As curved field: ... torus, no matter how one juggles the temperature and magnetic field

Thus total drift in a curved vacuum field is,

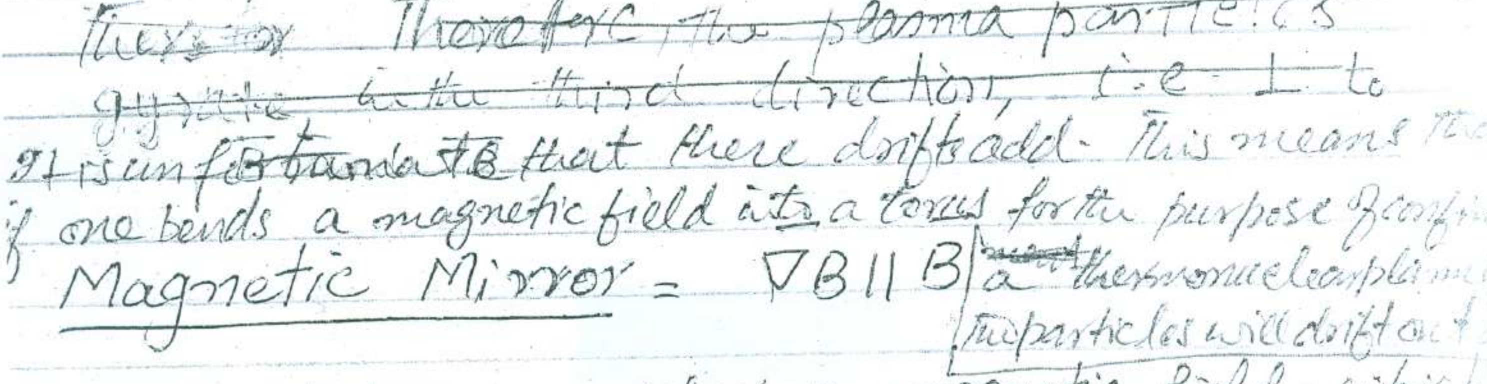
$$\vec{V}_R + \vec{V}_{\nabla B} = \frac{m \vec{R}_c \times \vec{B}}{q B^2 R_c^2}\left(V_\parallel^2 + \frac{1}{2} V_\perp^2\right)$$

~~Therefor Therefore, the plasma particles gyrate in the third direction, i.e. ⊥ to~~

It is unfortunate that these drifts add. This means that if one bends a magnetic field into a torus for the purpose of confining a thermonuclear plasma, the particles will drift out of

Magnetic Mirror = $\nabla B \parallel B$

Now we consider a magnetic field which is pointed primarily in the z-direction and whose magnitude varies in the z-direction. [Let the field be axisymmetric with $B_\theta = 0$ and $\frac{\partial}{\partial \theta} = 0$]



$B = \mu_0 N I$

No. of turns are different.
Magnetic field here is non-uniform.
This type of device is called magnetic mirror device.
The charged particle bounce back from the high field region.

$\bar{\nabla} \cdot \bar{B} = 0$ and $\boxed{\bar{\nabla} \cdot \bar{E} = 4\pi \rho_e}$

In cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{\partial B_z}{\partial z} = 0$$

We ignore $\frac{\partial B}{\partial \theta}$ because there is azimuthal symmetry in above picture. there is no angular dependance i.e. $B_\theta = 0$

$$\Rightarrow \int_0^r \frac{\partial}{\partial r}(r B_r)\, dr = -\int_0^r r \frac{\partial B_z}{\partial z}\, dr$$

$$r B_r = -\frac{1}{2} r^2 \left(\frac{\partial B_z}{\partial z}\right)_{r=0}$$

(1)

$$\Rightarrow B_r = -\frac{r}{2} \frac{\partial B_z}{\partial z}\bigg|_{r=0}$$

the Lorentz Force F

$$\bar{V} \times \bar{B} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ V_r & V_\theta & V_z \\ B_r & B_\theta & B_z \end{vmatrix}$$

$$\bar{F} = q(\bar{V} \times \bar{B})$$

$$F_r = q(V_\theta B_z - V_z B_\theta)$$

$$F_\theta = -q(V_r B_z - V_z B_r)$$

$$F_z = q(V_r B_\theta - V_\theta B_r)$$

As $B_\theta = 0$ and $\partial/\partial\theta = 0$.
there is no angular dependence.

$$\Rightarrow \quad F_r = q\, \underset{①}{v_\theta B_z}$$

$$F_\theta = q\,(-\underset{②}{v_r B_z} + \underset{③}{v_z B_r})$$

$$F_z = -q\, \underset{④}{v_\theta B_r}.$$

~~the last eqn tells us the particle gyrate and accelerated towards the z-axis and reflected back.~~

The terms ① and ② give rise to the usual Larmor gyration. Term ③ vanishes on the axis, when it does not vanish, this azimuthal force causes a drift in the radial direction. This drift merely makes the guiding centres follow the lines of force. Term ④ is the one we are interested in

$$F_z = -q\, v_\theta B_r$$

(tells us the particle gyrates & accelerates towards the z-axis and reflected back)

Using ① we have

$$F_z = -q v_\theta \frac{-r_L}{2} \frac{\partial B_z}{\partial z}$$

$$= \frac{1}{2} q v_\theta r_L \frac{\partial B_z}{\partial z}$$

To find $q v_\theta \frac{r_L}{2}$?

We know that when charged particle gyrate, the Larmour radius "r_L" and velocity are related with gyro frequency.

$$V_\perp = r_L \omega_c \quad (\text{and } V_\theta = V_\perp$$

$(v = r\omega)$

For simplicity, consider a particle whose guiding centre lies on the axis. Then V_θ is a constant during a gyration depending on the sign of q

$$V_\theta = \mp V_\perp$$

So

$$F_z = -\frac{1}{2} q V_\perp r_L \frac{\partial B_z}{\partial z}$$

$$= -\frac{1}{2} q V_\perp \frac{V_\perp}{\omega_c} \frac{\partial B_z}{\partial z}$$

$$= -\frac{1}{2} q \frac{V_\perp^2}{qB/m} \frac{\partial B_z}{\partial z}$$

$$= -\frac{1}{2} \frac{m V_\perp^2}{B} \frac{\partial B_z}{\partial z}$$

$\bar{\tau} = \bar{\mu} \times \bar{B}$

We define the magnetic moment of the gyrating particle to be;

$$\mu = \frac{1}{2} m V_\perp^2 / B$$

$$\Rightarrow \quad F_\parallel = -\mu \frac{\partial B_z}{\partial z}$$

This is a specific example of the force on

a diamagnetic particle, which in general can be written as,

$$F_{\parallel} = -\mu_{\parallel} \frac{\partial B_z}{\partial z}$$

$$F_{\parallel} = -\mu \nabla_{\parallel} B$$

Where dz is a line element along B.

→ Q: Prove μ is same as that of a current loop with area A and current I,

Ans: As $\mu = IA$



when a charged particle entered into it, then it will gyrate its current is

$$I = e/T$$

and $$A = \pi r_L^2 = \pi \frac{V_\perp^2}{\omega_c^2}$$

As $$\omega_c = 2\pi\nu = \frac{2\pi}{T}$$

$$\Rightarrow T = \frac{2\pi}{\omega_c} \Rightarrow I = \frac{e\omega_c}{2\pi}$$

~~$\mu = \frac{e2\pi}{\omega_c}$~~

$$\mu = \frac{e\omega_c}{2\pi} \times \pi \frac{V_\perp^2}{\omega_c^2}$$

$$I = \frac{e}{T} = \frac{e}{2\pi/\omega_c} = \frac{e\omega_c}{2\pi}$$

$$= \frac{1}{2} \cdot \frac{eV_\perp^2}{\omega_c}$$

$$= \frac{1}{2} \frac{e V_\perp^2}{eB/m} = \frac{1}{2} m V_\perp^2 / B$$

$$\mu = \frac{1}{2} \frac{m V_\perp^2}{B}$$ proved

Q - Prove μ is an invariant quantity,

Ans =

We know that

$$F_z = -\mu \frac{\partial B_z}{\partial z}$$

i.e. $$F_{\parallel} = -\mu \frac{\partial B_{\parallel}}{\partial z}$$

Magnetic field is parallel to z-component of force

$$m \frac{dv_{\parallel}}{dt} = -\mu \frac{\partial B_{\parallel}}{\partial z}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2} m v_{\parallel}^2\right) = -\mu \frac{\partial B_{\parallel}}{\partial z} v_{\parallel}$$

$$= -\mu \frac{\partial B_{\parallel}}{\partial z} \cdot \frac{dz}{dt}$$

$$= -\mu \frac{dB_{\parallel}}{dt}$$

$\frac{dB}{dt}$ is the time component of B observed by the particle.

As total energy of the particle is conserved so

$$\frac{d}{dt}\left(\frac{1}{2} m v_{\perp}^2 + \frac{1}{2} m v_{\parallel}^2\right) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2} m v_{\perp}^2\right) + \frac{d}{dt}\left(\frac{1}{2} m v_{\parallel}^2\right) = 0$$

Thus $\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) - \mu\frac{dB}{dt} = 0$

also $\mu = \frac{\frac{1}{2}mv_\perp^2}{B}$

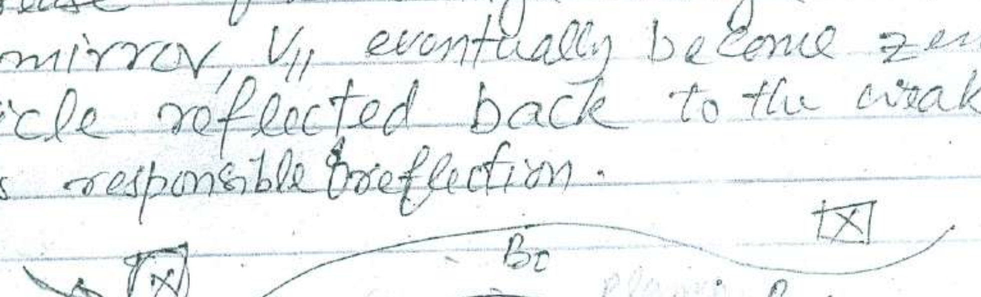
$$\Rightarrow \frac{d}{dt}(\mu B) - \mu\frac{dB}{dt} = 0 \quad \text{i.e. } \frac{d}{dt}(\mu B) = \mu\frac{dB}{dt}$$

$$\frac{d}{dt}(\mu B) = \Rightarrow \frac{d\mu}{dt} = 0$$

is known as the first adiabatic invariant of the particle orbit

μ the magnetic moment.

The invariance of μ is the basis for one of the primary schemes for plasma confinement, the magnetic mirror. As a particle moves from a weak-field region to a strong-field region in the course of its thermal motion, it sees an increasing B and therefore its $v_\perp$ must increase in order to keep μ constant. Since (*) its total energy must remain constant, $v_\parallel$ must necessarily decrease. If B is high enough in the "throat" of the mirror, $v_\parallel$ eventually become zero and the particle reflected back to the weak-field region. $F_\parallel$ is responsible to reflection.

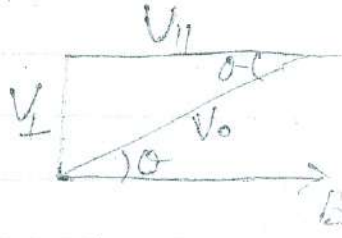


(1) Converging magnetic field.

(*) As E = 0 the total energy can not increase)

The trapping is not perfect, e.g. a particle with $V_\perp = 0$, will have no magnetic moment and will not feel any force along B. Also a particle with small $\frac{V_\perp}{V_\parallel}$ at the mid plane $B = B_0$ will also escape if the maximum field B_m is not large enough. A particle with $V_\perp = V_{\perp 0}$ and $V_\parallel = V_{\parallel 0}$ at the mid plane will have $V_\perp = V_\perp'$ and $V_\parallel = 0$ at its turning point. Let the field be B' there. Then invariance of μ gives,

$$\frac{1}{2} m V_{\perp 0}^2 / B_0 = \frac{1}{2} m V_\perp'^2 / B'$$



Conservation of energy requires,

$$V_\perp'^2 = V_{\perp 0}^2 + V_{\parallel 0}^2 = V_0^2$$



Thus we have

$$\frac{B_0}{B'} = \frac{V_{\perp 0}^2}{V_\perp'^2} = \frac{V_{\perp 0}^2}{V_0^2} = \sin^2\theta$$

$\sin\theta = \frac{V_\perp}{V_0}$

where θ is the pitch angle of the orbit in the weak field region. If θ is too small, B' exceeds B_m and particle does not mirror at all.

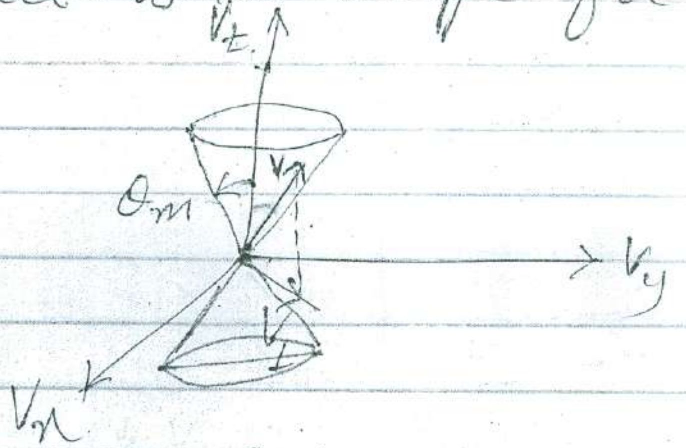
for $B' \to B_m$ .. The smallest θ of a confined particle is,

$$\sin^2\theta_m = B_0 / B_m = 1 / R_m.$$

where R_m is the mirror ratio.

The particles will be mirrored if their pitch angle in the weak field region ...

particles are confined in a mirror with ratio R_m if they have $\theta > \theta_m$: otherwise they escape. The above equation defines the boundary of a region in velocity space in the shape of a cone, called

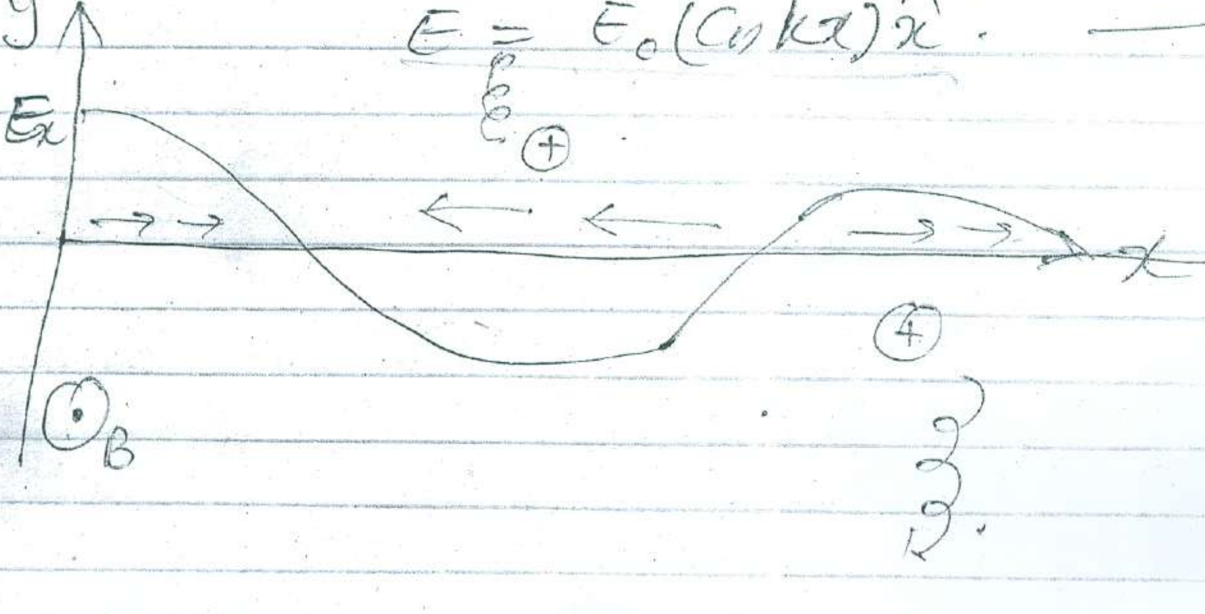


loss cone. Particles lying within the loss cone are not confined i.e. they can escape easily.

Non-Uniform $\bar{E}$-Field.

Now we let the magnetic field be uniform and the electric field be non-uniform. We assume E to be in the x direction and to vary sinusoidally in the x-direction,

$$\bar{E} = E_o(\cos kx)\hat{x}. \quad \text{——} (1)$$



Thus equation of motion is

$$m\frac{d\bar{v}}{dt} = q\left[\bar{E}(x) + \bar{v}\times\bar{B}\right]$$

$$\dot{v}_x = \frac{qB}{m}v_y + \frac{q}{m}E_x \quad \text{—— (2)}$$

$$\dot{v}_y = -\frac{qB}{m}v_x \quad \text{—— (3)}$$

$$\ddot{v}_x = -\omega_c^2 v_x \pm \omega_c \frac{\dot{E}_x}{B} \quad \text{—— (4)}$$

$$\ddot{v}_y = -\omega_c^2 v_y - \omega_c^2 \frac{E_x}{B} \quad \text{—— (5)}$$

E_x is the electric field at the position of the particle. ~~we want to calculate E_x.~~ we take undisturbed orbit to evaluate $E_x(x)$ and in the absence of E orbit is:

$$x = x_0 + r_L \sin\omega_c t \quad \text{—— (6)}$$

(5) $\Rightarrow$ $E_x = E_0 \cos k[x_0 + r_L \sin\omega_c t]$

$$\ddot{v}_y = -\omega_c^2 v_y - \frac{\omega_c^2}{B} E_0 \cos k\left[x_0 + r_L \sin\omega_c t\right] \quad \text{—— (7)}$$

Averaging over a gyration $\bar{v}_x = 0$

$\therefore$ $v_x = v_\perp \sin\omega_c t$, $\dot{E}_x = -E_0 k \sin\omega_c t$

The oscillating term $\ddot{v}_y$ averages to zero,

$$\Rightarrow \quad \ddot{v}_y = 0 = -\omega_c^2 \bar{v}_y - \omega_c^2 \frac{\overline{E_0}}{B}\overline{\cos k(x_0 + r_L \sin\omega_c t)}$$

$$0 = -\omega_c^2 \bar{v}_y - \omega_c^2 \frac{E_0}{B}\left(\cos kx_0 \cos(k r_L \sin\omega_c t) - \sin kx_0 \sin(k r_L \sin\omega_c t)\right)$$

and also by ~~Taylor's~~ Maclaurins' expansion,

$$\cos\epsilon = 1 - \frac{1}{\underline{2}}\epsilon^2 + \frac{\epsilon^4}{\underline{4}} - \frac{\epsilon^6}{\underline{6}}$$

$$\sin\epsilon = \epsilon - \frac{\epsilon^3}{\underline{3}} + \frac{\epsilon^5}{\underline{5}}$$

$$\Rightarrow \bar{V}_y = -\frac{E_0}{B_0}\left[\cos kx_0 \cdot \cos(kr_L \sin\omega_c t) - \sin kx_0 \cdot \sin(kr_L \sin\omega_c t)\right]$$

$$= -\frac{E_0}{B_0}\left[\cos kx_0\left(1 - \frac{1}{2}(kr_L \sin\omega_c t)^2 + \frac{(kr_L \sin\omega_c t)^4}{\underline{4}} + \ldots\right) - \sin kx_0\left((kr_L \sin\omega_c t) - \frac{(kr_L \sin\omega_c t)^3}{3!} + \ldots\right)\right]$$

Averaging over a period for cos-function is zero and also for cube and average over square is "1/2".

$$\Rightarrow \bar{V}_y = -\frac{E_0}{B}(\cos kx_0)\left(1 - \frac{k^2 r_L^2}{4}\right)$$

$$= -\frac{(E_0 \cos kx_0)}{B}\left(1 - \frac{k^2 r_L^2}{4}\right)$$

$$= -\frac{E_x}{B}\left(1 - \frac{k^2 r_L^2}{4}\right)$$

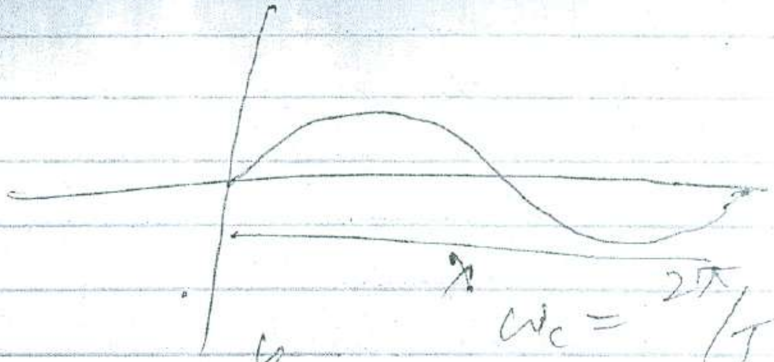
$$\bar{V}_E = \frac{\bar{E}\times B}{B^2}\left(1 - \frac{1}{4}k^2 r_L^2\right).$$

If we replace $\vec{\nabla}$ for ik Then

$$\bar{V}_E = \left(1 + \frac{1}{4}r_L^2 \nabla^2\right)\frac{\bar{E}\times B}{B^2}$$

The second term is called the finite Larmor-radius effect. Since r_L is much larger for ions than for electrons v_G is no longer independent of species. If a density clump occurs in a plasma an electric field can cause the ions and electrons to separate, generating another electric field. If there is a feedback mechanism that causes the second electric field to enhance the first ~~electromagnetic~~ electric field, $\bar{E}$ grows indefinitely and the plasma is unstable. Such an ~~(~~ instability called a drift instability.

clump - cluster



$$\int_0^{1/\omega_c} \cos 2\omega_c t \, dt$$

$$\left. \frac{\sin 2\omega_c t}{2\omega_c} \right|_0$$

$$\lambda = 2\pi/k$$

$$T =$$

$$\omega_c = 2\pi f$$

$$\omega_c = 2\pi/T$$

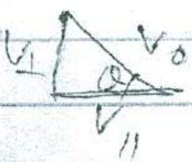
$$\Rightarrow \quad T = 2\pi/\omega_c$$

$\cos kx$

$\cos(kx + 2\pi)$

$\cos k(x + 2\pi/k)$

$$I = \frac{e\,\omega_c}{2\pi}$$



$$\frac{V_{\perp 0}^2}{V_0^2} = \sin^2\theta$$

